Can the derivative of an inverse equal the inverse of a derivative?

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When teaching a calculus class, it is standard to show that $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$ (as long as both expression make sense). At this point, it is usually also wise to emphasize that $(f^{-1})'(x)$ is not necessarily the same thing as $(f')^{-1}(x)$, and perhaps to do an example or two to demonstrate how these notationally-similar functions can behave quite differently. But a natural question arises: Are there any functions f so that $(f^{-1})'(x) = (f')^{-1}(x)$?

It's possible to cook up examples of functions f for which this equation holds at a single input. For example, if $f(x) = x^2$ then $(f')^{-1}(x) = \frac{x}{2}$ and $(f^{-1})'(x) = \frac{1}{2\sqrt{x}}$, so $(f')^{-1}(x) = (f^{-1})'(x)$ if and only if $x = 1$.

A more subtle question is whether the two *functions* can be equal, say along an open interval. The main result of this paper is that this is impossible!

Theorem 1. *Suppose f is a real-valued function so that $(f')^{-1}, (f^{-1})'$ are both defined on some interval (c, d). Then $(f')^{-1} \neq (f^{-1})'$ as functions on that interval.*

We prove this using elementary calculus. The main idea is to think about whether f is increasing or decreasing, and what that tells us about f' and f^{-1}. Then we look at the behavior of $(f')^{-1}$ and $(f^{-1})'$; if we assume that f is increasing on an interval, we can show that one of $(f')^{-1}$ and $(f^{-1})'$ is increasing while the other is decreasing; therefore they cannot be the same. The proof is a little more tricky when f is decreasing; in this case we analyze the domains and ranges of $(f')^{-1}$ and $(f^{-1})'$. We show that if $(f')^{-1} = (f^{-1})'$ along some interval, then f must be constant or increasing; both of these contradict the assumption that f is decreasing.

We also use minimal assumptions throughout this paper. For example, we will not assume that f'' exists (though we will take a brief detour to explore what concavity looks like when f'' is not defined). We will make great use of the Mean and Intermediate Value Theorems throughout.

Background

The proof of Theorem 1 will rely on a careful analysis of the behavior (increasing or decreasing) of $(f')^{-1}$ and $(f^{-1})'$. In this section we establish some necessary facts about real-valued functions.

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *strictly increasing* on (a, b) if $x < y$ implies $f(x) < f(y)$ for all $x, y \in (a, b)$. Similarly, f is *strictly decreasing* when $x < y$ implies that $f(x) > f(y)$, and f is *strictly monotone* if it is one of the two.

We begin by noting that the existence of $(f')^{-1}$ and $(f^{-1})'$ constrains the possible behavior of f and f'.

Theorem 2. *Let $a, b \in \mathbb{R}$ and suppose that $f : (a, b) \to \mathbb{R}$ so that $(f')^{-1}$ and $(f^{-1})'$ both exist. Then f and f' must be strictly monotone on (a, b).*

Proof. Since f is differentiable and invertible, it must be continuous and injective on (a, b). Since f is injective on an interval, f must be monotone.

To show that f' is strictly monotone requires a more careful argument; it is *a priori* possible that f' is not continuous, so f' being invertible alone does not imply that f' is strictly monotone. However, Darboux's Theorem says that if f is differentiable on an interval $[a, b]$ then f' satisfies the Intermediate Value Property on that interval; in other words, for any x between $f'(a)$ and $f'(b)$ there is some $c \in [a, b]$ so that $f'(c) = x$. Suppose that f' is invertible but not strictly monotone; then there are points $x < y < z$ so that (without loss of generality) $f'(z) > f'(x) > f'(y)$. Applying Darboux's Theorem to $[y, z]$ we see that there is some $w \in [y, z]$ so that $f'(w) = f'(x)$, which contradicts injectivity. So f' must be strictly monotone, as desired. ∎

Traditionally Darboux's Theorem is proven as a consequence of continuous functions achieving extrema on closed intervals; an alternative proof due to Olsen proves this with a clever application of the Mean Value Theorem [**2**].

It is not too hard to see (and it is illustrated in Figure 1) that the monotonicity of f and f' determine the monotonicity of f^{-1} and $(f^{-1})'$.

Theorem 3. *Let $a < b \in \mathbb{R}$ and let $f : (a, b) \to \mathbb{R}$ be invertible, differentiable, and strictly monotone. Assume further that f' is strictly monotone on (a, b).*

1. *If f is strictly increasing, then f^{-1} is strictly increasing. Furthermore, $(f^{-1})'$ has the opposite monotonicity of f'.*
2. *If f is strictly decreasing, then f^{-1} is strictly decreasing and $(f^{-1})'$ has the same monotonicity as f'.*

Proof. First we note that f and f^{-1} are both either strictly increasing or strictly decreasing. Indeed, suppose that $f(x)$ is strictly increasing. For any $s, t \in \operatorname{Range}(f)$ we know that $s < t$ if and only if $f^{-1}(s) < f^{-1}(t)$, so f^{-1} is strictly increasing. By an analogous argument, if f is strictly decreasing then so is f^{-1}.

Consider the case where f and f' are strictly increasing. Let $p, q \in (f(a), f(b))$ such that $p < q$. There exist $c, d \in (a, b)$ so that $p = f(c)$ and $q = f(d)$. Since f is strictly increasing, $c < d$. Since f' is strictly increasing, $f'(c) < f'(d)$. By the formula for the derivative of inverse, we have

$$(f^{-1})'(p) = (f^{-1})'(f(c)) = \frac{1}{f'(c)} > \frac{1}{f'(d)} = (f^{-1})'(f(d)) = (f^{-1})'(q).$$

Therefore $(f^{-1})'$ is strictly decreasing, so f^{-1} is concave down. The other three cases follow by analogous arguments. ∎

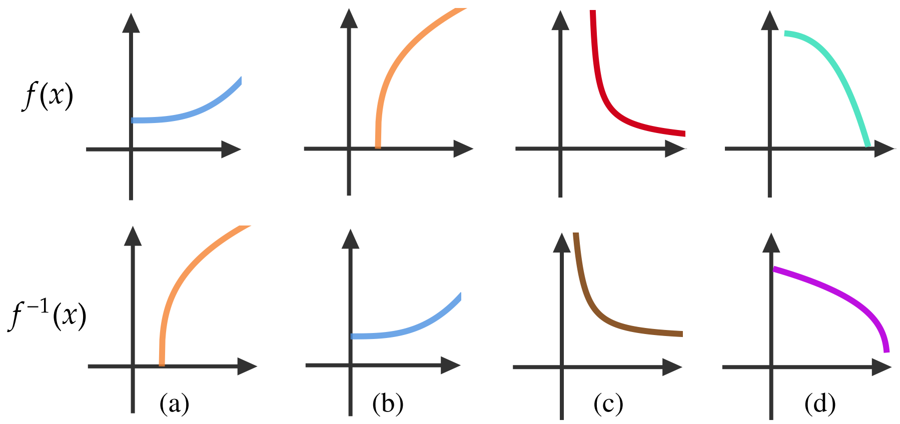

Figure 1. The examples show the four possible combinations of monotonicity for f and f', and the corresponding behavior of f^{-1}.

An aside on concavity of non-differentiable functions

The condition of monotonicity of f' in Theorem 3 can be interpreted naturally as a concavity condition. In this section we set aside our motivating question for a moment and explore this connection. In particular, we prove an analogue of Theorem 3 without the assumption that f is differentiable. Our proofs will rely primarily on applications of the Mean and Intermediate Value Theorems, and we include them here primarily for their educational interest.

Intuitively, a function is concave up when it “curves” upward, while it is concave down when it “curves” downward. If a function f is twice differentiable, we can define its concavity by analyzing f'' as follows: a function f is concave up if $f''(x) > 0$, and concave down if $f''(x) < 0$. With this definition of concavity, we can see that f is concave up exactly when f' is increasing, and concave down exactly when f' is decreasing. In fact, [**5**] use this condition exactly to define the concavity of a function. Equivalently, [**4**] defines concavity of a differentiable function by comparing the function to its tangent lines.

However, it is also possible to define concavity of a function f without assuming that f is differentiable, as in [**1, 3**], by comparing the function to its secant chords. To emphasize the concavity definition we are using, we will call this *chord convexity*. A function is *chord concave* up on some interval when every secant chord on that interval lies above the graph of f. To be more precise, we introduce some notation for the secant chord of f between s and t: Let $C^f_{s,t}(x)$ denote the linear function whose graph is the line segment from $(s, f(s))$ to $(t, f(t))$.

Definition. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let (a, b) an interval. We say that f is *chord concave up* on (a, b) if for any $s, t \in (a, b)$, $C^f_{s,t}(x) > f(x)$ for all $x \in (s, t)$. We say that f is *chord concave down* on (a, b) if for any $s, t \in (a, b)$, $C^f_{s,t}(x) < f(x)$ for all $x \in (s, t)$. The function has *fixed chord concavity* on (a, b) if it is chord concave up or chord concave down on (a, b).

In fact, all of these possible definitions of concavity are equivalent whenever they are defined. In particular, we will use the Mean Value Theorem to show that when f is differentiable, the “chord concavity” of f is determined by monotonicity of f'. Alternative proofs of this fact may be found in [**3**] Appendix to Chapter 11, Theorems

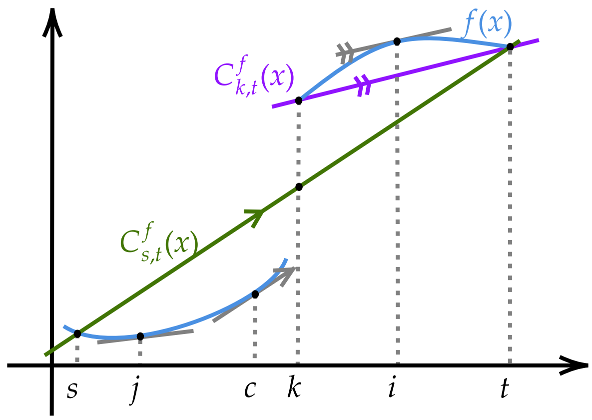

Figure 2. If f' is strictly increasing, then f is chord concave up.

1 and 2 or in [**1**] Chapter 2, Theorem 2.9. The proof in [**3**] first shows that the result is true under the additional assumption that $f(a) = f(b)$, and then extends to the more general case. The proof included here does not take that intermediate step. The proof in [**1**] compares concavity of $A(x) = \int_a^x f(t)dt$ and monotonicity of f. It may be helpful to refer to Figure 2 while reading the proof.

Theorem 4. *Let f be a differentiable function on (a, b). Then f is chord concave up on (a, b) if and only if f' is strictly increasing. Similarly, f is chord concave down on (a, b) if and only if f' is strictly decreasing.*

Proof. Suppose f' is strictly increasing on the interval (a, b). Let $s < t \in (a, b)$ and let m be the slope of $C^f_{s,t}$. Suppose (for contradiction) that there is some $k \in (s, t)$ so that $f(k) \geq C^f_{s,t}(k)$. By the Mean Value Theorem, there exists a $c \in (s, t)$ such that $f'(c) = m$. Hence for any $j \in (s, c)$ we know $f'(j) < m = f'(c)$. Then

$$f(j) = f(s) + \int_s^j f'(x)\,dx < f(s) + \int_s^j m = tC^f_{s,t}(j).$$

Thus if $f(k) \geq C^f_{s,t}(k)$, then $k \in [c, t)$. Let n be the slope of $C^f_{k,t}(x)$. Note that $m \geq n$ since

$$m = \frac{f(t) - C^f_{s,t}(k)}{t - k} \geq \frac{f(t) - f(k)}{t - k} = n.$$

By the Mean Value Theorem, there exists $i \in (k, t)$ such that $f'(i) = n$. However, since $f'(c) = m \geq n = f'(i)$ this contradicts the assumption that f' is strictly increasing. Thus for all $k \in [c, t)$ we must have $f(k) < C^f_{s,t}(k)$. Therefore f is chord concave up.

On the other hand, suppose that f is chord concave up. Pick any $s < t \in (a, b)$. We know that $C^f_{s,t}(x) > f(x)$ for all $x \in (s, t)$. Thus

$$f'(s) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \leq \lim_{h\to 0^+} \frac{C^f_{s,t}(x+h) - f(x)}{h} = (C^f_{s,t})'(x).$$

By an analogous computation, $(C^f_{s,t})'(x) \leq f'(t)$, so $s < t$ implies that $f'(s) \leq f'(t)$. If $f'(s) = f'(t)$ then $f'(x) = f'(s)$ for all $x \in (s, t)$. Thus f is a line, and

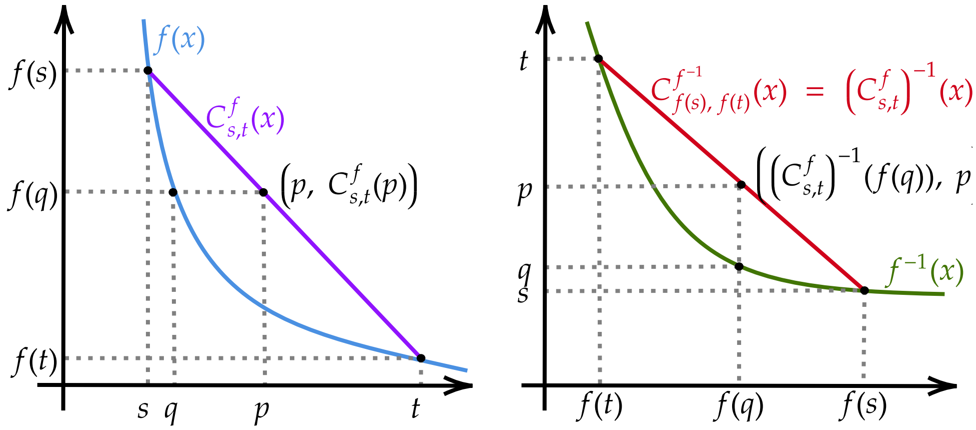

Figure 3. If f is strictly decreasing and chord concave up, then f^{-1} is strictly decreasing and chord concave up.

in particular it must be the line $y = C^f_{s,t}(x)$. But then f is not concave up, so $s < t$ implies that $f'(s) < f'(t)$, and f' is strictly increasing.

An analogous argument shows that f' strictly decreasing if and only if f is chord concave down. ■

Together with Theorem 3 this shows that if f is differentiable, then the (chord) concavity of f^{-1} is dependent on the monotonicity and chord concavity of f. However, we can prove the same result without the assumption that f is differentiable directly from the definition of chord concavity. The proof uses the Intermediate Value Theorem as a key tool. (This is illustrated in Figure 3.)

Theorem 5. *Let $a < b \in \mathbb{R}$ and let $f : (a, b) \to \mathbb{R}$ be invertible, continuous, strictly monotone, and of fixed chord concavity on (a, b).*

1. *If f is strictly increasing, then f^{-1} has the opposite chord concavity of f.*
2. *If f is strictly decreasing, then f^{-1} has the same chord concavity as f.*

Proof. There are four cases, depending on the chord concavity and monotonicity of f. We will prove this in the case that f is strictly decreasing and chord concave up. The other three cases follow a similar argument.

Suppose that f is strictly decreasing and chord concave up on (a, b). Since f is a bijection from (a, b) to $(f(b), f(a))$, it suffices to show that for any $s, t \in (a, b)$, we have $C^{f^{-1}}_{f(s),f(t)}(x) > f^{-1}(x)$ for all $x \in (f(t), f(s))$. Choose arbitrary $s, t \in (a, b)$ and $p \in (s, t)$. Then $C^f_{s,t}(p) > f(p)$. Notice that $f(s) = C^f_{s,t}(s) > C^f_{s,t}(p) > f(p)$, so by the Intermediate Value Theorem there is some $q \in (s, p)$ so that $C^f_{s,t}(p) = f(q)$. Then $(C^f_{s,t})^{-1}(f(q)) = p$. Notice that $(C^f_{s,t})^{-1}$ is a line going through $(f(s), s)$ and $(f(t), t)$. Since $C^{f^{-1}}_{f(s),f(t}$ is a line going through the same points, we have $(C^f_{s,t})^{-1}(f(q)) = C^{f^{-1}}_{f(s),f(t)}(f(q)) = p$. But $f^{-1}(f(q)) = q < p$ so $(C^{f^{-1}}_{f(s),f(t)})(x) > f^{-1}(x)$ for all $x \in (f(t), f(s))$. Thus f^{-1} is concave up. ■

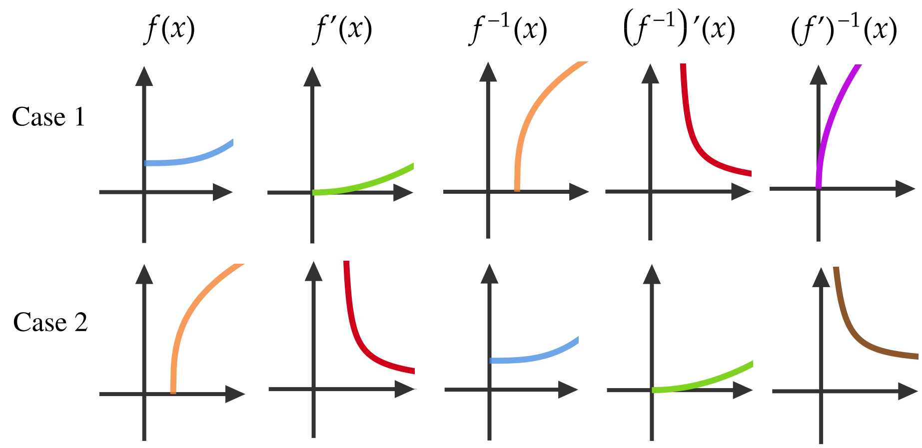

Figure 4. Examples of strictly increasing functions f with fixed concavity are shown, along with f', f^{-1}, $(f')^{-1}$ and $(f^{-1})'$.

Proof of the Main Theorem

We now return to Theorem 1. We will proceed by considering cases. Since f and f' must be strictly monotone for $(f')^{-1}$ and $(f^{-1})'$ to be defined, there are four cases to consider:

CASE 1: f is strictly increasing and f' is strictly increasing,
CASE 2: f is strictly increasing and f' is strictly decreasing,
CASE 3: f is strictly decreasing and f' is strictly increasing, and
CASE 4: f is strictly decreasing and f' is strictly decreasing.

The proofs for Case 1 and 2 are similar, and rely on showing that $(f^{-1})'$ and $(f')^{-1}$ have opposite monotonicities. It may be helpful to refer to Figure 4.

Proof of Theorem 1; Cases 1, 2. Suppose that $f : (a, b) \to \mathbb{R}$ is a function so that $(f')^{-1}$, $(f^{-1})'$ are both defined on some interval (c, d).

CASE 1. Consider the case where f and f' are strictly increasing. Then f^{-1} is defined on the interval (c, d). By Theorem 3, $(f^{-1})'$ must be strictly decreasing on this interval. Since f' is strictly increasing in this case, $(f')^{-1}$ is also strictly increasing by Theorem 3. Since $(f')^{-1}$ is strictly increasing and $(f^{-1})'$ is strictly decreasing, $(f')^{-1} \neq (f^{-1})'$.

CASE 2. By an analogous argument, if f is strictly increasing and f' is strictly decreasing then $(f')^{-1}$ must be strictly decreasing while $(f^{-1})'$ is strictly increasing. ■

One might hope to prove Cases 3 and 4 in an analogous way. However, in these cases $(f')^{-1}$ and $(f^{-1})'$ have the same monotonicity. Thus the proofs for Case 3 and 4 will be a little more subtle. For both cases we proceed by contradiction. If $(f')^{-1} = (f^{-1})'$, then the two functions must have the same domain and range. In Case 3 we show that this implies that f' is constant, which contradicts our assumptions. In Case 4 we show that this implies that $f(x) = \ln x + C$ for some constant C, which also contradicts our assumptions on f.

Proof of Theorem 1; Cases 3, 4. Suppose that $f : (a, b) \to \mathbb{R}$ is a function so that $(f')^{-1}$, $(f^{-1})'$ are all defined on some interval (c, d). Assume for contradiction that

$(f')^{-1} = (f^{-1})'$ on some interval (c, d).

CASE 3 Consider the case where f is strictly decreasing and f' is strictly increasing. Note that f decreasing implies that the range of f is $(f(b), f(a))$, so the domain of $(f^{-1})'$ is $(f(b), f(a))$. Up to replacing (a, b) with a sub-interval, we may assume that $c = f(b), d = f(a)$. Now, since f' is strictly increasing, the range of f' is $(f'(a), f'(b))$. Therefore the domain of $(f')^{-1}$ is $(f'(a), f'(b))$. Since $(f')^{-1} = (f^{-1})'$, we must have $c = f'(a) = f(b)$. Note that if $(f')^{-1} = (f^{-1})'$ on $(f(b), f(a))$, then they are also equal on any interval of the form $(f(x), f(a))$ for $a < x < b$. By applying the same argument, we get $f'(a) = f(x) = f(b)$ for all $x \in (a, b)$. Since f is strictly increasing this is a contradiction.

CASE 4 Consider the case where f and f' are strictly decreasing. Following a similar argument as in Case 3, we can see that the range of $(f')^{-1}$ is (a, b) and the range of $(f^{-1})'$ is $\left(\frac{1}{f'(a)}, \frac{1}{f'(b)}\right)$. Since $(f^{-1})' = (f')^{-1}$ they must have the same range, so $b = \frac{1}{f'(b)}$. Let $a < x < b$. By applying the same argument to the interval (a, x), we get $f'(x) = \frac{1}{x}$. Since x is arbitrary, this must be true for all $x \in (a, b)$. Therefore $f(x) = \ln x + C$ for some constant C. Since $\ln x + C$ is strictly increasing, this is a contradiction. ■

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Summary. The formula for the derivative of the inverse is a standard calculus computation, but the difference between $(f^{-1})'$ and $(f')^{-1}$ is often a challenge for students to keep track of. In this paper we explore whether it is possible to find a function f so that these two functions are equivalent. We show that there is no such function using basic ideas familiar to any calculus student, like monotonicity. Along the way we take a short detour to explore what concavity means when a function is not twice differentiable, and demonstrate some slick applications of the Mean and Intermediate Value Theorems.

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