TOPIC PAPER: CUBULATED GROUPS

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1. Introduction

This paper is an introduction to cubulated groups, and their application to Agol's proof of the Virtual Haken Conjecture. It begins with a review of necessary concepts from δ-hyperbolic geometry and its applications to groups. Section 3 defines cube complexes. We explore their relationship with Right Angled Artin Groups and describe special cube complexes in terms of hyperplanes, local isometries, and finally separability of quasiconvex subgroups. In Section 4 we define a cubulated group, introduce Sageev's construction, and give some examples. Section 5 is then devoted to a return to 3-manifolds, and in particular giving a sketch of the tools used to prove the Virtual Haken Conjecture.

2. δ-Hyperbolic Geometry and Hyperbolic Groups

This section will provide some of the necessary background on geometric group theory for the rest of the paper. Theorems from this section come from [2], unless otherwise stated.

The field of δ-hyperbolic geometry is the study of (usually geodesic) metric spaces which are "negatively curved in the large." The model examples are trees and hyperbolic space of fixed dimension.

Example (The Hyperbolic Plane)**.** The upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ with the metric $ds_{\mathbb{H}} = \frac{ds_{\mathbb{E}}}{y}$ is a model for the hyperbolic plane, called the *upper half-space model*. The group $PSL(2, \mathbb{R})$ acts by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \mapsto \frac{az+b}{cz+d},$$

which induce isometries of $\mathbb{H}$. Conversely, every orientation-preserving isometry of $\mathbb{H}$ arises this way.

The geodesics in this model are vertical straight lines and arcs of circles (in the Euclidean metric) perpendicular to $\mathbb{R}$.

Now let $G = \langle S|R\rangle$ a finitely presented group. Denote the Cayley graph of G with respect to this presentation as $C_S(G)$. We endow this graph with the word metric. Note that the action of G on itself by left multiplication gives an isometry of $C_S(G)$ to itself, and this action is properly discontinuous, and free if G has no 2-torsion.

A priori, the structure of the Cayley graph depends greatly on the choice of presentation of G. However, given any two presentations of G, the map of Cayley graphs $C_S(G) \to C_{S'}(G)$ induced by the identity map on G is a quasi-isometry. More generally, we have the following lemma.

Theorem 2.1 (Schwartz Lemma)**.** *Let G act properly discontinuously and cocompactly by isometries on a proper geodesic metric space X. Then G is finitely generated by some set S, and the orbit map $G \to X$ sending $g \mapsto gx$, for some $x \in X$, is a quasi-isometry from $C_S(G)$ to X.*

Example. Let G a group and H a finite index subgroup of G. Then H acts properly discontinuously by isometries and cocompactly on $C(G)$, so G and H are quasi-isometric.

We will now formalize the idea of "negatively curved in the large."

Definition (δ-Hyperbolic Space)**.** A geodesic metric space (X, d) is *δ-hyperbolic* if geodesic triangles are δ-thin.

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Returning to the example of the upper half-plane model of $\mathbb{H}$, one can see that geodesic triangles are δ-thin, where $\delta = \log(1+\sqrt{2})$. This can be seen by comparison triangles. We can formalize this process as follows:

Definition (CAT(K) Space)**.** A geodesic metric space X is CAT(K) for some K if triangles are thinner than comparison triangles in a space of constant curvature K.

Notice that if $K < 0$ and X is CAT(K), then X is δ-hyperbolic for some δ.

The Schwartz lemma may make one think that working with δ-hyperbolic spaces and quasi-isometries is sufficient, but the image of a geodesic under a quasi-isometry is not in general a geodesic. However, such an image, called a *quasigeodesic*, is uniformaly close to an actual geodesic.

Theorem 2.2 (Morse Lemma)**.** *Let (X, d_X) be a proper δ-hyperbolic space. Then for any K, ϵ there is a constant $C = C(\delta, K, \epsilon)$ such that any (K, ϵ)-quasigeodesic γ is within Hausdorff distance C of a genuine geodesic γ^g. If γ has two endpoints, we may take γ^g to have the same endpoints.*

Corollary 2.3. *Let $f : X \to Y$ a quasi-isometry, and Y δ-hyperbolic. Then X is δ'-hyperbolic for some δ'.*

In fact, being quasigeodesic is a local condition in a hyperbolic space. We say that a path γ in X is a *k-local geodesic* if every subpath of length $\leq k$ is a geodesic.

Lemma 2.4. *Let X be a δ-hyperbolic, and let $k > 8\delta$. Then any k-local geodesic is a (K, ϵ)-quasigeodesic, for some K and ϵ depending on k and δ.*

Note the Morse Lemma shows that if $C_S(G)$ is hyperbolic, then so is every other Cayley graph of G. This motivates the following fundamental definition:

Definition (Hyperbolic Group)**.** A group G is *hyperbolic* if $C_S(G)$ is δ-hyperbolic for some finite generating set S and some δ.

Example. Free groups are hyperbolic, as their Cayley graphs are trees.

Example. Fundamental groups of closed orientable surfaces of genus at least 2 are hyperbolic.

Finally, we introduce a concept which will become useful in future sections.

Definition (Quasiconvex)**.** A subset Y of a geodesic metric space X is *quasiconvex* if there exists a $k \geq 0$ such that every geodesic segment with endpoints in Y lies entirely within the k-neighborhood of Y.

To apply this to groups, one wants to say that a subgroup $H \leq G$ is quasiconvex if H is a quasiconvex subspace of $C_S(G)$ for some S. A priori, this depends on the choice of presentation of G. However, in the case that G is hyperbolic, H being quasiconvex in $C(G)$ is not dependent on choice of presentation, as quasiconvexity is preserved by quasi-isometry in δ-hyperbolic spaces.

Furthermore, one can verify that:

(1) finite and finite index subgroups are quasiconvex,
(2) a quasiconvex subgroup is finitely generated,
(3) quasiconvexity is preserved by intersection, and
(4) a subgroup is quasiconvex if and only if it is finitely generated and quasi-isometrically embedded.

3. Cube Complexes

3.1. **Introduction.** This section introduces the notion of a cube complex. References are from [18] unless otherwise stated.

An *n-cube* is a copy of $[-\frac{1}{2}, \frac{1}{2}]^n$. A *face* of an n-cube is a restriction of some of its coordinates to $\pm\frac{1}{2}$. A *cube complex* is a cell complex obtained by gluing together n-cubes along their faces. Note that this is entirely combinatorial.

Definition (NPC cube complex, CAT(0) cube complex)**.** A cube complex is CAT(0) if it is CAT(0) as a metric space. It is *nonpositively curved*, or NPC, if its universal cover is CAT(0).

This is not always easy to identify. In fact, there is a nice combinatorial condition which is equivalent to a cube complex being NPC. A *flag complex* is a complex in which every complete subgraph spans a simplex. Note that a graph is flag if and only if the length of its shortest cycle is ≥ 4. The following was first identified by Gromov [6]. The proof follows [3].

Theorem 3.1 (Flag Condition)**.** *A cube complex is NPC if and only if the link of every vertex is a flag complex.*

Proof. This condition is necessary since the boundary of a 'quadrant' is not convex. On the other hand, a complex is locally CAT(0) if the links of vertices are CAT(1). The flag condition is inherited by passing to links so by induction, we have reduced to the case of a 2-dimensional complex. Hence links are metric graphs made of segments of length $\pi/2$. So every loop in these links has length at least 2π, so there are no atoms of positive curvature in the 2-dimensional complexes, and they are CAT(1) if spherical or NPC if cubical. □

Example. Trees are CAT(0), and graphs are NPC. A pillowcase (i.e. the identification of two 2-cubes along their boundaries) is not NPC.

Example (Graphs of graphs)**.** Let X a topological space that decomposes as a graph Γ of spaces where each vertex space X_v is a graph, and each edge space $X_e \times [-1,1]$ is the product of a graph and an interval. Suppose the attaching maps $\phi_{e-} : X_e \times \{-1\} \to X_{\iota(e)}$ and $\phi_{e+} : X_e \times \{1\} \to X_{\tau(e)}$ are combinatorial immersions. Then X is an NPC square complex.

Example (Amalgams along $\mathbb{Z}$)**.** A group of the form $F_n *_{\mathbb{Z}} F_n$ is isomorphic to the fundamental group of a graph of spaces where each vertex space X_v is a bouquet of n circles. Indeed, up to conjugation we can think of our group as $\langle a_1, \ldots, a_n \rangle *_{u=v} \langle b_1, \ldots, b_n \rangle$, where u and v are cyclically reduced words in the a_i and b_i, respectively. Then u, v can be thought of as immersed combinatorial paths in the corresponding bouquets of circles, and the group thus arises as the fundamental group of a graph of spaces where each edge space is a cylinder attached along the cycles u and v. By subdividing the vertex spaces so that $|u| = |v|$, the cylinder can be divided into squares. By the above example, this is an NPC cube complex.

Example. The following comes from [15]. Suppose that G is a group acting on a graph X, and let T the subgraph of X containing vertices p, q and edge $e = (p, q)$. Suppose that T is a fundamental domain of $G \backslash X$. Let G_p, G_q, G_e the stabilizers of the vertices and edge of T, respectively. Then the following are equivalent:

(1) X is a tree.
(2) The homomorphism $G_p *_{G_e} G_q \to G$ induced by the inclusions $G_p \to G$ and $G_q \to G$ is an isomorphism.

Conversely, one also sees that if $G = G_1 *_A G_2$, then there is a tree X (unique up to isomorphism) on which G acts, with fundamental domain a segment as above, with stabilizers as above.

3.2. Right Angled Artin Groups.

Definition (RAAG)**.** Let Γ be a graph. The *right-angled Artin group*, or RAAG, associated to Γ, denoted $G(\Gamma)$ is a group with one generator for each vertex of Γ and relations that two generators commute if and only if the corresponding vertices in Γ are joined by an edge. In other words,

$$G(\Gamma) = \langle g_v : v \in V(\Gamma) \mid [g_v, g_w] : (v, w) \in E(\Gamma) \rangle.$$

There is a canonical NPC cube complex associated to each RAAG, called the *Salvetti complex*, and denoted $R(\Gamma)$. This is constructed as follows: take the standard 2–complex of $G(\Gamma)$ from the presentation above. Add an n-cube for each collection of n pairwise commuting generators (equivalently, glue an n-cube along every complete subgraph of size n). Since this satisfies the flag condition, this complex is certainly NPC.

Theorem 3.2. *For every (finite) simplicial graph Γ, there is a (compact) NPC cube complex $R(\Gamma)$ such that the associated RAAG $G(\Gamma)$ can be identified with $\pi_1(R(\Gamma))$.*

Notice that $R(\Gamma)$ has a single 0-cell, v, and that the link of v contains two copies of Γ. Also, the 2-skeleton of $R(\Gamma)$ is precisely equal to the presentation complex.

RAAGs have several very nice properties, including the following, from [4]:

Theorem 3.3. *RAAGs are residually torsion-free nilpotent.*

Proof. Let G the RAAG associated a graph Γ. Let V denote the vertex set of Γ, and E the edge set of Γ. Then $\mathbb{Z}[[V]] := \{\sum^{\infty} n_w w \mid w \text{ is a word in V}\}$. Let I the two-sided ideal of $\mathbb{Z}[[V]]$ generated by $\{xy - yx \mid (x, y) \in E\}$, and then define $\mathbb{Z}[[\Gamma]] := \mathbb{Z}[[V]]/I$, and let ϕ the natural surjection $\mathbb{Z}[[V]] \to \mathbb{Z}[[\Gamma]]$. Let U the group of units of $\mathbb{Z}[[\Gamma]]$, and define a group homomorphism $\theta : G \to U$ by

$$\theta : x \mapsto 1 - x, \qquad x^{-1} \mapsto 1 + x + x^2 + \ldots$$

for every vertex x. One can check that this map is injective.

Now define a collection of subgroups $J_n \leq U$ where J_n is the collection of elements of the form $1 + c$, where c is a linear combination of monomials of degree at least n. Then J_n is normal in U. So define a homomorphism $\theta_n : G \to U/J_n$ as the composition

$$\theta_n : G \xrightarrow{\theta} U \to U/J_n.$$

Let $G_n = \ker \theta_n$.

Then one can check that G/G_n is torsion-free for each n, G_n/G_{n+1} is in the center of G/G_{n+1}, and G/G_n is nilpotent.

Finally, note that $\bigcap J_n = \{1\}$, so $\bigcap G_n = \{1\}$. So for any non-trivial $g \in G$, there is some G_n such that $g \notin G_n$, and G/G_n is torsion-free nilpotent. □

In addition, Hsu-Wise proved in [8] the following.

Theorem 3.4. *RAAGs embed in Coxeter groups.*

Proof. Let Γ a graph with vertex set V and edge set E. Define a Coxeter group C generated by $\{x_v, y_v v \in V\}$ by the relators:

$$order(x_v, y_w) = order(y_v, x_w) = \begin{cases} \infty & \text{if } v = w, \\ 2 & \text{if } v \neq w, (v,w) \in E, \\ \infty & \text{if } v \neq w, (v,w) \notin E. \end{cases}$$

$$order(x_v, x_w) = order(y_v, y_w) = \begin{cases} 2 & \text{if } v = w, \\ 2 & \text{if } v \neq w, (v,w) \in E, \\ \infty & \text{if } v \neq w, (v,w) \notin E. \end{cases}$$

One can check that the homomorphism $G(\Gamma) \to C$ given by $g_v \mapsto x_v y_v$ is an injection. □

Example. Recall that all Coxeter groups are linear. Indeed, let C a Coxeter group defined by a symmetric $n \times n$ matrix $M = (m_{ij})$. Define a real vector space V with basis $E = \{e_1, \ldots, e_n\}$, and let B denote the symmetric linear form on V given by $B(e_i, e_j) = -2\cos(\pi/m_{ij})$. Note that if $m_{ij} = \infty$, then $B(e_i, e_j) = -2$. Now define $\rho_i : V \to V$ by

$$\rho_i : x \mapsto x - B(x, e_i)e_i.$$

One can check that for each i, B is ρ_i-invariant and $\rho_i \rho_j$ has order m_{ij}.

So for C, with generating set denoted by $\{r_1, \ldots, r_n\}$, define a linear representation $\rho : C \to GL(V)$ by

$$\rho : r_{i_1} \cdots r_{i_k} \mapsto \rho_{i_1} \cdots \rho_{i_k}.$$

One can prove that this is faithful, hence C is linear.

There is an immediate corollary of the previous Theorem and Example:

Corollary 3.5. *RAAGs are linear.*

3.3. Hyperplanes.

Definition (Hyperplanes)**.** A *midcube* of an n-cube is the intersection of the cube with a coordinate plane $x_i = 0$. An edge is said to be *dual* to a midcube it they intersect. A *hyperplane* of a cube complex X is a maximal connected subspace that intersects each cube in a single midcube or the empty set.

Note that each edge is dual to a unique hyperplane. There is an equivalence relation on edges of X, where two edges are equivalent if they are dual to the same hyperplane.

Hyperplane behavior and subgroups give much of the richness of the theory of cube complexes. Part of this structure comes from the nice behavior of hyperplanes in CAT(0) complexes, and more generally in *special cube complexes* (described in a future section).

Given hyperplanes H and H' in a cube complex X, we say that two hyperplanes *osculate* if there are distinct edges $e, f \in X^1$, which do not span a square such, that e is dual to H and f is dual to H', and e and f share a vertex. Note that we may assume that $H = H'$, in which case we say that H *self-osculates.* If H is 2-sided, then we can give an orientation to each dual edge of H. We say that H *directly* self-osculates if e and f share a terminal or initial point, and *indirectly* self-osculates if the terminal point of e is the initial point of f (or vice versa). We say two distinct hyperplanes *interosculate* if they both intersect and osculate.

Theorem 3.6 (Hyperplanes in CAT(0) Cube Complexes, see [12])**.** *Let $\tilde{X}$ a CAT(0) cube complex. Then the following are true:*

1. *Each midcube lies in a unique hyperplane of $\tilde{X}$.*
2. *Immersed hyperplanes do not self-cross or self-osculate (directly or indirectly).*
3. *A hyperplane H of $\tilde{X}$ is itself a CAT(0) cube complex.*
4. *A hyperplane is 2-sided, and the cubical neighborhood $N(H) \cong H \times [-\frac{1}{2}, \frac{1}{2}]$ is a convex subcomplex of $\tilde{X}$ called the carrier of H.*
5. *$\tilde{X} - H$ has two components.*

In this context, a *geodesic* is an edge path in $\tilde{X}^1$ which is geodesic in the graph metric, and a subcomplex $Y \subset \tilde{X}$ is *convex* if for every geodesic that begins and ends in Y, the entire geodesic lies in Y. The proofs of these facts are highly combinatorial, and make use of *disk diagrams.* The key is to look at disks with 'corners' and apply combinatorial Gauss-Bonnet.

Definition (Disk Diagram)**.** A *disk diagram D* is a compact, contractible, combinatorial 2-complex with a chosen planar embedding $D \subset \mathbb{R}^2$. Its *boundary path*, ∂D, is the attaching map of the 2-cell containing the point at ∞. A *diagram in a complex X* is a combinatorial map $D \to X$.

The main tool of the proof is the idea of *bigon removal.* If $D \to X$ is a disk diagram in an NPC cube complex X, and there is some subdiagram B of D corresponding to a bigon of dual curves in X, then B can be replaced through homotopy by a subdiagram B' of smaller area. Indeed, choose a minimal area bigon within D, and a lowest triangle of dual curves with at least one side on the bigon. Then by a hexagon move we can push the front three squares to the back three squares. We repeat inductively until we reach a cancellable pair, where we can decrease the area of the bigon B by two squares.

Additionally, through the existence of *cornsquares*, a square s such that the dual curves which cross in s end in adjacent squares along the boundary of D, we can also prove the following result, relating hyperplanes and geodesics.

Theorem 3.7. *Let γ a geodesic in a CAT(0) cube complex $\tilde{X}$, and H a hyperplane of $\tilde{X}$. Then γ intersects H at most once.*

Proof. Suppose γ is a curve which intersects each hyperplane at most once. Then it is geodesic, since any other curve γ' would have to cross those hyperplanes, as hyperplanes split $\tilde{X}$.

On the other hand, suppose γ passes through two edges e and f dual to the same hyperplane H. Then we can shorten its length by 2, contradicting that it is geodesic. Indeed, consider a minimal area disk diagram D between the subpath of γ joining e and f, and a path $\sigma \to N(H)$ with the same endpoints. Then there is a cornsquare in D with an outerpath on σ, and we can use hexagon moves to move the cornsquare to s' along σ. Then notice that s' lies on $N(H)$, so D was not minimal area. □

3.4. **Local Isometries.** Recall that for complexes $A \subset B$, A is *full* if for all $x, y \in A$, if x, y are adjacent in B, then they are also adjacent in A.

Definition (Local Isometry)**.** A combinatorial map of NPC cube complexes, $\phi : Y \to X$, is a *local isometry* if for each $y \in Y^0$, the induced map $\text{lk}(y) \to \text{lk}(\phi(y))$ is injective and $\text{lk}(y)$ embeds as a full subcomplex.

Heuristically, a map is a local isometry if it is locally injective and has no "missing squares."

Example. Any covering map is a local isometry. It is of course locally injective, and surjectivity implies that there are no "missing squares."

Example. Let $\phi : Y \to X$ a local isometry. Then $\phi : \tilde{Y} \to \tilde{X}$ is an embedding as a convex subcomplex. This was first noted in [9].

Theorem 3.8. . *Let H a hyperplane of a CAT(0) cube complex $\tilde{X}$, and let $N(H)$ its carrier. Then $N(H)$ is a convex subcomplex of $\tilde{X}$.*

Proof. Since H neither self-osculates nor self-intersects, there are no "corners" in $N(H)$. Hence this inclusion is a local isometry. □

3.5. **Special Cube Complexes.**

Definition (Special Cube Complex)**.** An NPC cube complex is *special* if:

1. every hyperplane embeds,
2. every hyperplane is 2-sided,
3. no hyperplane directly self-osculates, and
4. no two hyperplanes interosculate.

Example. By comments above, any CAT(0) cube complex is special.

Example. Any subcomplex of a product of two graphs is a special cube complex.

Example. Any Salvetti complex $R(\Gamma)$ is a special cube complex. Indeed, we have already shown that $R(\Gamma)$ is NPC. Futhermore, it is a subcomplex of some (very high dimensional) n-torus. So every hyperplane is a subcomplex of a hyperplane of an n-torus, and are therefore embedded and 2-sided. Furthermore, each hyperplane is dual to a single edge, so it can't self-osculate. If two hyperplanes cross, then their dual edges must span a square, so there is no interosculation.

In fact, this example is stronger than it first appears.

Theorem 3.9. *An NPC cube complex X is special if and only if there exists some local isometry $\phi : X \to R$, where R is the Salvetti complex of some graph.*

Proof. Suppose that $A \to B$ is a local isometry and B is special. Then A is special, since the four pathologies are preserved by local isometries.

On the other hand, let Γ the *crossing graph* of X. The vertex set of Γ corresponds to the hyperplanes of X, and two vertices are adjacent if the corresponding hyperplanes cross. As hyperplanes are 2-sided, we can consistently direct the 1-cells of X which are dual to the same hyperplane. Then this gives us a map $X^1 \to R(\Gamma)$, which extends to a local isometry $X \to R$. No self-osculation implies that it is an immersion, and no interosculation implies that it is a local isometry. □

At this point, we have a topological and a combinatorial description of special cube complexes. We can also describe them with an algebraic condition, if we broaden to *virtual* specialness. A group is said to be *virtually P* (for some property P) if it has a finite index subgroup which is P. Similarly, a complex is *virtually P* if it has a finite cover which is P.

Recall that if G is a group and H a subgroup, then H is *separable* in G if for all $g \in G - H$, there is a finite index subgroup $K \leq G$ such that $H \leq K$ and $g \notin K$. Notice that a subgroup is separable if and only if it is the intersection of finite index subgroups.

Scott [14] related subgroup separability to finite covers of spaces in the following theorem.

Theorem 3.10. *Let $\hat{X} \to X$ a covering map of complexes. Then $\pi_1(\hat{X})$ is separable in $\pi_1(X)$ if and only if for every compact subcomplex $K \subset \hat{X}$, there exists an intermediate cover $\overline{X} \to X$ such that $\hat{X} \to X$ factors as $\hat{X} \to \overline{X} \to X$, and $K \to \overline{X}$ is an embedding.*

As a direct corollary, we can see:

Corollary 3.11. *Suppose Y is compact, $Y \to X$ is π_1-injective, and Y lifts to an embedding in a the cover $\hat{X}$ associated to $\pi_1(Y)$, and $\pi_1(Y)$ is separable in $\pi_1(X)$. Then $Y \to X$ lifts to an embedding in a finite cover.*

Using these tools, we can prove the following result.

Theorem 3.12. *Let X a compact NPC cube complex. Then X has a finite special cover if and only if*

1. *$\pi_1(U)$ is separable in $\pi_1(X)$ for all hyperplanes U in X, and*
2. *$\pi_1(U)\pi_1(V)$ is separable in $\pi_1(X)$ for all pairs of crossing hyperplanes U, V.*

Proof. (Sketch). Suppose the two conditions hold. Then if U is a hyperplane, there is a finite cover in which U is embedded (i.e. not self-intersecting), 2-sided, and not directly self-osculating. Similarly, if U and V are crossing hyperplanes, then there is a finite cover in which they do not interosculate. Since the cube complex is compact, there are finitely hyperplanes, and so there is a finite cover which simultaneously satisfies all of these conditions. This is the finite special cover. □

Putting the concepts of specialness and local isometries together, we can prove the following:

Theorem 3.13 (Canonical Completion and Retraction). *Let X a special cube complex and Y a compact NPC cube complex, and let $\phi : Y \to X$ a local isometry. Then there is a finite cover $C = C(X \to Y)$ such that ϕ lifts to an embedding $\hat{\phi} : Y \to C$ and C retracts to the image of Y.*

The idea is to construct such a cover through *fiber products*. Recall that if $\alpha : A \to R$ and $\beta : B \to R$ are local isometries of cube complexes, the fiber product $A \otimes_R B$ is the cube complex whose n-cubes are pairs of n-cubes in A and B respectively that map to the same cube in R. Notice that $A \otimes_R B$ is not necessarily connected! But if α and β are connected covering maps, then the component of $A \otimes_R B$ containing the 0-cell (a, b) is the based cover that is the smallest common cover of the based covers (A, a) and (B, b) of (R, r), where $\alpha(a) = \beta(b) = r$.

So given X a special cube complex, we have a local isometry to some special (Salvetti) cube complex R. Since we have a local isometry $Y \to X$, the composition gives a local isometry $Y \to R$. Consider the induced map of 1-skeletons, $Y^1 \to R^1$. Note R^1 is a bouquet of circles. We can complete Y^1 to 2-cover of R^1, $C(Y^1 \to R^1)$. Note that Y^1 embeds in $C(Y^1 \to R^1)$, and there is a canonical retraction map. By specialness, we can extend this to a covering map $C(Y \to R)$. Then we form the cover $C = C(Y \to X)$ as the fiber product $C = C(Y \to R) \otimes_R X$. The retraction map $C \to Y$ is given by composition.

Using this, we get a quick proof of the following:

Theorem 3.14. *Suppose X is special and compact, and $\pi_1(X)$ is hyperbolic. Then every quasiconvex subgroup H of $G = \pi_1(X)$ is separable.*

Proof. Let $\sigma \notin H$. Then there is an H-cocompact convex subcomplex $\tilde{Y}$ containing $\tilde{\sigma}$. Let $Y = H\backslash\tilde{Y}$. Let $G' = \pi_1(C)$, where C is the (finite) cover constructed above. Then $H \leq G'$, but $\sigma \notin G'$. □

4. Cubulated Groups

This section is primarily taken from [3].

Definition (Cubulated Group). A group G is *cubulated* if it admits a properly discontinuous cocompact action by isometries on some CAT(0) cube complex.

Definition (Ends of Groups, Relative Ends). Let G a finitely generated group. The *ends* of G are the ends of a Cayley graph of G with respect to a finite generating set. The number of ends is denoted $e(G)$. Let H a subgroup of G. The *ends of G relative to H* are the ends of $H\backslash C(G)$. The number of relative ends is denoted $e(G, H)$.

Furthermore, there is a famous result of Stallings [16] which characterizes groups and their ends:

(1) $e(G)$ is one of $0, 1, 2, \infty$;
(2) $e(G) = 0$ if and only if G is finite;
(3) $e(G) = 2$ if and only if G is virtually $\mathbb{Z}$;
(4) $e(G) = \infty$ if and only if G splits non-trivially as a graph of groups with finite edge groups.

Note that $e(G, H) = 0$ if and only if H is finite index in G. If G splits non-trivially over H, then $e(G, H) > 1$. But the converse is not true in general.

Example. The following is from [15]. Suppose $G = A * H$. Note first that $e(G, H) = \infty$, or both A and H have order 2 and $e(G, H) = 1$. Indeed, consider the tree associated the to splitting. H acts on this tree as a vertex stabilizer, and distinct H-orbits of ends of the tree correspond to distinct ends of G relative to H.

Then G splits over H if and only if A is a non-trivial free product (and so $G = (A_1 * H) *_H (H * A_2)$) or A is infinite cyclic. Indeed, if $G = X *_H Y$, then since no conjugate of A meets H, A freely decomposes into the intersection of itself with the conjugates of X and Y.

So let $A \cong H = \langle a, b, c, d \mid b^a = b^2, c^b = c^2, d^c = d^2, d^a = a^2 \rangle$. This is finitely generated infinite simple. Let $G = A * H$. Then A is indecomposable and not infinite cyclic, so G does not split over H, even though $e(G, H) = \infty$.

Definition (Codimension-1 Subgroup). Let H a subgroup of a finitely generated group G. If $e(G, H) > 1$, then H is a *codimension 1 subgroup*.

Example (Sageev's Construction)**.** The following comes from [13]. Given a collection of codimension 1 subgroups H_i of a group G, one can construct a CAT(0) cube complex on which G acts. Suppose first there is a single such subgroup, H. Fix a generating set of G, and let $C(G)$ the associated Cayley graph. There is some compact subset of $H\backslash C(G)$ which separates $H\backslash C(G)$ into two unbounded sets. Let B be one of these unbounded components, and A the preimage of B in $C(G)$. Note that A is H-invariant, and if $g \in G$ the symmetric difference $A\Delta Ag$ intersects finitely many H-orbits.

Now let Σ the set of (left) translations of A and A^c. A *vertex* V is a subset of Σ such that the following hold:

(1) Exactly one of X or X^c is in V, for all X in Σ.
(2) If $A \in V$ and $A \subset B$, and $B \in \Sigma$, then $B \in V$.

Construct a graph Γ' with this vertex set, where there is an edge (V, W) if there exists some $X \in \Sigma$ such that $W = (V - X) \cup X^c$.

There is a special vertex for each $g \in G$, defined as $V_g = \{X \in \Sigma \mid g \in X\}$. Take the subgraph Γ which consists of each connected component of Γ' containing a vertex V_g for any $g \in G$. This is the one skeleton our desired CAT(0) complex. Note that it is connected since any two vertices of the form V_g are connected. Indeed, if V and V' have a finite symmetric difference $S = \{X_1, X_2, \ldots, X_n\}$, then any $A_i \in V$ which is minimal in S is also minimal in V. Then $(V - A_i) \cup A_i^c$ is adjacent to V and has a smaller symmetric difference with V', and so by induction V and V' are in the same component. But also every vertex V_g and V_h have a finite symmetric difference.

Glue in the higher dimension cubes by a flag construction, i.e. glue an n-cube to every complete graph on n vertices. This gives a CAT(0) cube complex with a natural G-action.

This generalizes to the case of a collection of codimension-1 subgroups H_i in G by defining Σ to be the collection of left translates of A_i and A_i^c, where A_i is an unbounded set associated to each H_i. Vertices are edges are defined analagously.

Theorem 4.1. *Sageev's construction gives a CAT(0) cube complex.*

Proof. This is NPC by the flag condition. One can see that it is simply connected by finding a shortest closed curve which is non-trivial. Each edge is labeled with some minimal set $X_i \in \Sigma$. By definition of the vertices, there must be a first pair $i > j$ such that $X_i = X_j^c$. One can then slide across a square spanned by X_j and X_{j+1} to get a new curve of the same length. Repeat this process, and eventually this gives a cancellation, shortening the length of the path, which is a contradiction. ☐

Now we want this action to be proper and cocompact. To see that this action is cocompact requires a lemma:

Lemma 4.2 (Quasiconvex Helly's Theorem)**.** *Let G a hyperbolic group and fix some $K > 1$. Then for any integer k and for any R, there is an R' so that if $L_1, \ldots, L_k$ are K-quasiconvex subsets of G with $N_R(L_i) \cap N_R(L_j)$ non-empty for all i, j, then $\bigcap_i N_{R'}(L_i)$ is non-empty.*

Using this, we can prove the following:

Theorem 4.3. *Let G hyperbolic, and H_i a finite family of codimension-1 quasiconvex subgroups of G. Then the Sageev constructions gives a finite-dimensional CAT(0) cube complex X, and G acts on X cocompactly.*

To see that this action is proper requires further conditions.

Theorem 4.4 (Bergeron-Wise Boundary Criterion)**.** *Let G be word hyperbolic. Suppose that for each pair of distinct points p, q in $\partial_\infty G$ there is a quasiconvex codimension-1 subgroup H such that p and q are in distinct components of $\partial_\infty G - \partial_\infty H$. Then there is a finite collection H_i of quasiconvex subgroups such that the action of G on the corresponding CAT(0) cube complex X is proper and cocompact.*

Example. Let M a closed hyperbolic 3-manifold. Then a theorem of Kahn-Markovic says that given $\epsilon > 0$, let $p \in M$ and $v \in UT_pM$. Then there is a $(1 + \epsilon)$-quasigeodesic closed immersed surface S in M passing through p and perpendicular to v. Then $\pi_1(S)$ is a quasiconvex, codimension-1 subgroup of $\pi_1(M)$, and since p and v can be chosen arbitrarily, we can separate pairs of points in $\partial_\infty \pi_1(M)$. Hence there is a CAT(0) cube complex associated to a sufficiently large collection of subgroups of the form $\pi_1(S)$, and $\pi_1(M)$ acts properly and cocompactly on this complex. Since $\pi_1(M)$ is torsion-free, the action is free. Thus $\pi_1(M)$ is isomorphic to the fundamental group of an NPC cube complex.

Example ($C'(1/6)$ groups)**.** Let $G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_2 \rangle$, where we assume the r_i to be cyclically reduced. A *piece* of a subword of some r_i which is also a subword of another $r_j^{\pm}$, or appears elsewhere as a subword of r_i. Then let G satisfy the $C'(1/6)$ condition, i.e. for every piece σ in some r_i, $length(\sigma)/length(r_i) < 1/6$. Let K the Cayley 2-complex. One can cubulate such a group as follows. Assume each r_i has even length. Then build a graph X which has vertices the 1-cells of K, and edges connecting each vertex to its antipodal point across a disk in K. Note that this also gives X an embedding in K. Since the presentation is $C'(1/6)$, the components of X give rise to codimension-1 subgroups of $\pi_1(K) = G$, and there are enough of them to use the Sageev construction to cubulate.

Example (Random group of density less than $\frac{1}{6}$, See [10])**.** A *random group at density d* is a group G with presentation $G = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_s \rangle$, where $s = (2k-1)^{nd}$ and the relators are all cyclically reduced words of length n. One says that a property holds for a random group of density d *with overwhelming probability*, or (wop), if the property holds with probability going to 1 as $n \to \infty$.

Gromov proved several facts about random groups, including

(1) at density $d > 1/2$, random groups are either trivial or $\mathbb{Z}/2\mathbb{Z}$ (wop),
(2) at density $0 < d < 1/2$, random groups are hyperbolic (wop), and
(3) at density d, a random group satisfies $C'(2d)$, but not $C'(\lambda)$ for any $\lambda < 2d$ (wop).

Using essentially the same wall construction as in the previous example, Ollivier-Wise proved that at density $d < 1/6$ a random group is cubulated (wop). They show that these walls are in fact trees, and that they can be used to define codimension-1 subgroups. This is more subtle than the $C'(1/6)$ case, as is showing that there are enough to cubulate.

5. The Virtual Haken Theorem

5.1. **Background.** For this section, we will assume that all 3-manifolds are oriented and connected. Recall that a compact 3-manifold M (possibly with boundary) is *irreducible* if every embedded 2-sphere in M bounds a 3-ball in M. Notice that by the sphere theorem, this is equivalent to $\pi_2(M)$ being trivial. A properly embedded surface S in M is *incompressible* if for every embedded disk D in M which intersects S in a loop, the boundary ∂D bounds a disk in S. Similarly, S is *boundary incompressible* if for every embedded disk D in M which intersects S in a proper arc (with the rest of ∂D on ∂M), the arc $\partial D \cap S$ is homotopically inessential in S. A surface which is incompressible and boundary incompressible is *essential*.

Definition (Haken)**.** A compact 3-manifold M is *Haken* if it is irreducible and it contains an essential properly embedded subsurface.

Example. Suppose M is an irreducible 3-manifold, and that $\pi_1(M)$ acts minimally and without inversions on a tree T. There is such an action if and only if $\pi_1(M)$ can be expressed as a nontrivial amalgam $A *_B C$, or an HNN extension $A*_B$. The quotient of the action is a graph Γ, so we can build an equivariant map $\tilde{M} \to T$. The preimage of a regular value is then an embedded surface in M, which may be compressed until it is essential. Since M is irreducible, M is Haken.

Conversely, the Loop Theorem implies that an incompressible embedded surface S in M is π_1-injective. Thus one may cut along S to express $\pi_1(M)$ as a non-trivial amalgam or HNN extension over a surface subgroup. Thus, for incompressible 3-manifolds, Hakenness is equivalent to the existence of a nontrivial splitting of $\pi_1(M)$.

Example. Let M an irreducible manifold with boundary. Then either M is a ball, or ∂M has no spherical components. The intersection pairing on homology defines a symplectic structure on $H_1(\partial M)$, and the kernel of the inclusion $H_1(\partial M) \to H_1(M)$ is a Lagrangian subspace, and is therefore not the whole space. Hence the rank of $H_1(M)$ is positive.

Then by duality, there are nontrivial integral classes in $H^1(M)$ and $H_2(M, \partial M)$. A properly embedded surface S which represents $\alpha \in H^1(M, \partial M)$ is Thurston norm minimizing if it has no sphere or disk components and minimizes $-\chi(S)$ among all surfaces in α. Such surfaces exist, and they must be essential, because otherwise they can be compressed, which would reduce $-\chi(S)$. Thus M is Haken.

Notice that a Haken manifold can be cut along an essential surface to produce a new Haken manifold (with boundary). Haken showed that after finitely many such iterations, the result can be taken to be a collection of manifolds of the form $S \times [0,1]$, where S is a surface. This process is called a *hierarchy*, and opens the door to proving things about Haken manifolds with inductive proofs.

In 1968, Waldhausen [17] conjectured the following:

Theorem 5.1 (Virtual Haken Conjecture)**.** *Every aspherical closed 3-manifold has a finite cover which is Haken.*

Perelman proved that if M is an aspherical closed 3-manifold which does not contain an essential torus, then M is either small Seifert fibered (and hence Haken), or M is hyperbolic. This reduced proving Waldhausen's conjecture to the case that M is closed and hyperbolic.

Waldhausen's conjecture was proved to be true by Agol, using the tools of cube complexes. The remainder will describe some of the specific tools used by Agol in his proof.

5.2. **Moving Towards a Proof.** The heart of Agol's proof lies in the following theorem:

Theorem 5.2 (Agol [1])**.** *Let G a hyperbolic group which acts properly and cocompactly on a CAT(0) cube complex $\tilde{X}$. Then G has a finite index subgroup G' such that $\tilde{X}/G'$ is a special cube complex.*

Indeed, recall that every closed hyperbolic 3-manifold has a fundamental group G which is isomorphic to the fundamental group of some compact NPC cube complex X. Let $\tilde{X}$ its universal cover. Then by the above theorem, G has a finite subgroup G' such that $\tilde{X}/G'$ is a special cube complex. Hence G is virtually special. Since surface subgroups are quasiconvex, they are separable, and so M is Haken by Scott's theorem.

This theorem is easily stated, but very difficult to prove. One important tool used in the proof is the concept of *relative hyperbolicity*. To understand this, we need a combinatorial horoball. Let Γ a simplicial graph with vertex set V. The *combinatorial horoball* associated to Γ, denoted $\mathcal{H}(\Gamma)$, is the graph with vertex set $V \times \mathbb{N}$ and two types of edges:

(1) a *vertical edge* from (v,i) to $(v,i+1)$ for every $v \in V$, and
(2) a *horizontal edge* from (v,i) to (w,i) whenever $d_\Gamma(v,w) < 2^i$.

Note that $\mathcal{H}(\Gamma)$ is δ-hyperbolic, and any two vertices in $\mathcal{H}(\Gamma)$ can be joined by a *model geodesic*, consisting of at most two vertical paths and a horizontal path of length at most 3.

Now let G be a finitely generated group, and let P a finitely generated subgroup. Let S a finite generating set of G such that $S \cap P$ is a finite generating set of P. Then the Cayley graph $C_{S\cap P}(P)$ is a subgraph of $C_S(G)$. By attaching a horoball to each (left) translate of $C_{S\cap P}(P)$ in $C_S(P)$, we obtain a new graph, denoted $X(G,P,S)$, on which G acts isometrically (but not co-compactly). Analogously, for any finite collection of finitely generated subgroups, $\mathcal{P} = \{P_1,\dots,P_n\}$, we can attach a horoball for each P_i to obtain a graph $X(G,\mathcal{P},S)$, as long as $S \cap P_i$ is a finite generating set of P_i.

Definition (Relatively Hyperbolic)**.** Let G a finitely generated group, and let $\mathcal{P} = \{P_1,\dots,P_n\}$ a finite collection of finitely generated subgroups of G. Then $(G,\mathcal{P})$ is *relatively hyperbolic* if the space $X(G,\mathcal{P},S)$ is hyperbolic.

Definition (Almost malnormal)**.** A collection of subgroups $\mathcal{P} = \{P_i\}$ of a group G is said to be *almost malnormal* if for all $g \in G$, the intersection $P_i^g \cap P_j$ if infinite only when $i = j$ and $g \in P_i$.

Remark. A theorem of Bowditch proves that if G is hyperbolic and $\mathcal{P}$ is a collection of almost malnormal quasiconvex subgroups of G, then G is hyperbolic relative to $\mathcal{P}$.

Let G be hyperbolic relative to $\mathcal{P} = \{P_i\}$, and let $N_i \trianglelefteq P_i$ be a normal subgroup of P_i, for each i. Let $\mathcal{N} = \langle\langle N_i \rangle\rangle$ be the normal subgroup of G generated by the N_i. Then $G/\mathcal{N}$ is the result of *Dehn filling* along the N_i.

Theorem 5.3 (Groves-Manning [7], Osin [11]; Hyperbolic Dehn Surgery Theorem)**.** *Suppose G is hyperbolic relative to $\mathcal{P}$. Then there is a finite subset A of $G - \{e\}$ such that if no N_i meets A, then*

(1) *the natural map $P_i/N_i \to G/\mathcal{N}$ is injective for all i, and*
(2) *the group $G/\mathcal{N}$ is hyperbolic relative to the $\{P_i/N_i\}$.*

Furthermore, if F is any finite subset of G we can choose A as above so that $\phi : G \to G\mathcal{N}$ is injective on F, and $\phi(F) \cap \phi(P_i) = \phi(F \cap P_i)$ for all i.

Notice the similarity to Thurston's well-known theorem about Dehn fillings of cusped hyperbolic 3-manifolds. The method of proof is similar to proofs of small cancellation theory, and involves inspecting Van Kampen diagrams and linear isoperimetric inequalities.

Notice also the if the P_i are residually finite, we can always find N_i which satisfy the criteria. We say a Dehn filling is *peripherally finite* if each N_i is finite index in P_i. So again, if each P_i is residually finite, then we can find a peripherally finite Dehn filling which is hyperbolic.

Theorem 5.4 (Agol-Groves-Manning [1]; Weak Separation)**.** *Let G be a hyperbolic group, and let H a quasiconvex subgroup of G which is isomorphic to the fundamental group of a virtually special NPC cube complex, and let $g \in G - H$. Then there is a group G' and a surjection $\phi : G \to G'$ such that*

1. *G' is hyperbolic,*
2. *$\phi(H)$ is finite, and*
3. *$\phi(g) \notin \phi(H)$.*

Note that if we could take G' to be finite, then H would actually be separable. It ends up that it suffices to have that $\phi(H)$ is finite. The proof of this theorem inducts on the height of a group.

Definition (Height)**.** Let G be hyperbolic and H quasiconvex in G. The *height* of H is the least integer n so that if there are elements $g_1, \ldots, g_n \in G$ such that $H, H^{g_1}, \ldots, H^{g_n}$ are distinct, then the intersection of the conjugates $H \cap H^{g_1} \cap \cdots \cap H^{g_n}$ is infinite.

Notice that H has height 0 if and only if H is finite, and H has height 1 if and only if it is almost malnormal and infinite.

Lemma 5.5 (Finite Height, [5])**.** *Every quasiconvex subgroup of a hyperbolic group has finite height, and for any quasiconvex H there are only finitely many H-conjugacy classes of infinite groups of the form $H \cap H^{g_1} \cap \cdots \cap H^{g_n}$.*

The idea behind the proof of the Weak Separation Theorem is to let $\mathcal{P}$ be a collection of H-conjugacy classes of minimal infinite intersections of the form $H \cap H^{g_1} \cap \cdots \cap H^{g_k}$. These are quasiconvex. Then replace each $P \in \mathcal{P}$ with its commensurator, $P' = \{g \in G \mid P \cap P^g \text{ is finite index in } P \text{ and } P^g\}$. Since P is quasiconvex in H, P is finite index in P'.

Choose an element P' of each H-conjugacy class to get a new collection, $\mathcal{P}'$. Then each $P' \in \mathcal{P}'$ is almost malnormal in H, so $(H, \mathcal{P}')$ is relatively hyperbolic. Similarly, replace each P' with its commensurator in G, P'', and choose an element of each G-conjugacy class to get a collection $\mathcal{P}''$. Since $\mathcal{P}''$ is almost malnormal in G, $(G, \mathcal{P}'')$ is relatively hyperbolic.

The aim is to find $N_i \trianglelefteq P_i''$ to do a Dehn filling, in such a way that we simultaneously get a Dehn filling of $(G, \mathcal{P}'')$ and $(H, \mathcal{P}')$.

Definition (H-filling)**.** A collection $N_i \trianglelefteq P_i''$ gives rise to an *H-filling* if whenever $P_j' \cap (P_i'')^g$ is infinite for some $P_j' \in \mathcal{P}$, then N_i^g is contained in P_j'.

Now such a set N_i gives rise to a Dehn filling of $(G, \mathcal{P}'')$. A G-conjugacy class of P_i'' gives rise to several H-conjugacy classes $P_{i,j}'$. So if $\{N_i\}$ gives rise to an H-filling, then $N_i \trianglelefteq P_{i,j}'$ for each j, and thus also induces a Dehn filling of $(H, \mathcal{P}')$.

Proposition 5.6 (Agol-Groves-Manning [1])**.** *Let G hyperbolic, and $H \leq G$ quasiconvex such that the height of H is at least 1. Let $(G, \mathcal{P}'')$ and $(H, \mathcal{P}')$ as above, and $g \in G - H$. Then for all sufficiently long peripherally finite H-fillings $\phi : G \to G/\mathcal{N}$,*

1. *$\phi(H)$ is isomorphic to the result of the induced filling of H,*
2. *$\phi(H)$ is quasiconvex in $G/\mathcal{N}$,*
3. *$\phi(g) \notin \phi(H)$, and*
4. *the height of $\phi(H)$ in $G/\mathcal{N}$ is strictly less that the height of H in G.*

So to prove the Weak Separability theorem, one would like to use this proposition to induct on the height. The problem is that we do not know if H admits perpherally finite fillings, since the P_i'' may not be residually finite. The P_i'' are virtually subgroups of H, so it suffices to prove that H is residually finite. In the statement of the Weak Separability Theorem, H is the fundamental group of a virtually special NPC cube complex, so H is residually finite in the first step. To induct, we need that $\phi(H)$ is also residually finite.

Theorem 5.7 (Wise [19]; Malnormal Special Quotient Theorem)**.** *Let G be hyperbolic, and let $\mathcal{P} = \{P_i\}$ be a family of almost malnormal and quasiconvex subgroups so that $(G, \mathcal{P})$ is relatively hyperbolic. Suppose G is the fundamental group of a virtually special NPC cube complex. Then there are finite index subgroups P_i' in the P_i so that if $\phi : G \to G(N_1, \ldots, N_m)$ is any peripherally finite filling with each N_i contained P_i', then $\phi(G)$ is the fundamental group of a virtually special NPC cube complex.*

We will take the Malnormal Special Quotient Theorem (MSQT) as a black box, but mention that the method of proof relies on inductively built hierarchies.

Definition ($\mathcal{QVH}$)**.** A hyperbolic group G has a *quasiconvex virtual hierarchy*, or G is in $\mathcal{QVH}$, if it is obtained inductively by the following operations:

(1) The trivial group is in $\mathcal{QVH}$.
(2) If $G = A *_B C$, where $A, C \in \mathcal{QVH}$ and B is finitely generated and quasiconvex in G, then G is in $\mathcal{QVH}$.
(3) If $G = A*_B$ where $A \in \mathcal{QVH}$ and B is finitely generated and quasiconvex in G, then G is in $\mathcal{QVH}$.
(4) If H is finite index in G and H is in $\mathcal{QVH}$, then G is in $\mathcal{QVH}$.

Example. Free groups are in $\mathcal{QVH}$, as are fundamental groups of closed oriented surfaces with genus at least 2.

Example. A Haken manifold has a fundamental group with is in $\mathcal{QVH}$ if its fundamental group is hyperbolic and the decomposing surfaces subgroups at each step of the hierarchy are quasiconvex. In particular, a closed hyperbolic 3-manifold with an embedded essential surface that is not the fiber of a fibration has a fundamental group in $\mathcal{QVH}$.

The main reason to study this hierarchy is that it precisely captures when a group is virtually special. The following is proven in [19] for torsion-free groups, and in the general case in the appendix of [1].

Theorem 5.8. *A hyperbolic group acts cocompactly on a CAT(0) cube complex with special quotient if and only if it is in $\mathcal{QVH}$.*

The proof of this follows from a similar theorem about *malnormal quasiconvex virtual hierarchies* and the MSQT.

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